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Rational Contractions in $\mathcal{G}$ -Metric Spaces and Fixed Point Results

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Abstract: Present paper demonstrate some fixed point results for rational contractions in $\mathcal{G}$-metric space. Our outcomes extend and generalize various famous results in the literature.

1. INTRODUCTION

Fixed point theory has been an exciting research area for long. It is exceedingly developed and still prospering under a number of new domains. Encouraged by the fact that metric space theory has wide-ranging applications not only in mathematics but in the other area of quantitative sciences, numerous researchers have focused extensive attention to extend the concept of a metric space. In 2006, Mustafa and Sims[3] initiated an innovative description of the generalized metric by introducing $\mathcal{G}$-metric spaces. Literature on $\mathcal{G}$-metric space has developed a lot in recent time and so many fixed points results on $\mathcal{G}$-metric space appeared [1-2, 4-7]. We are presenting some fixed points results of rational contraction in $\mathcal{G}$ -metric space.

2. PRELIMINARIES

In 2006, Mustafa and Sims[3] initiated an innovative description of the generalized metric by introducing $\mathcal{G}$-metric spaces. They revealed that most of the outcomes relating to D-metric space are unacceptable. Literature on $\mathcal{G}$-metric space has developed a lot in recent time and so many fixed points results on $\mathcal{G}$-metric space appeared.

Definition 2.1 [3] A mapping $\mathcal{G}: \Pi \times \Pi \times \Pi \rightarrow [0, \infty)$ on a non empty set Π, satisfies the following properties for every $p, q, r, \alpha \in \Pi$:

(G-1) $\mathcal{G}(p, q, r) = 0$ if and only if $p = q = r$,

(G-2) $0 < \mathcal{G}(p, p, q)$, with $p \neq q$,

(G-3) $\mathcal{G}(p, p, q) \leq \mathcal{G}(p, q, r)$, with $q \neq r$, (G-4) $\mathcal{G}(p,q,r) = \mathcal{G}\{\pi(p,r,q)\}$, where π is a permutation,

(G-5) $\mathcal{G}(p,q,r) \leq \mathcal{G}(p,\alpha,\alpha) + \mathcal{G}(\alpha,q,r)$.

The function $\mathcal{G}$ is described as generalized metric on Π and the pair $(\Pi,\mathcal{G})$ is known to be $\mathcal{G}$-metric space.

Example 2.2[3] Consider $(\Pi,\mathcal{G})$ be a metric space and mapping $\mathcal{G}: \Pi \times \Pi \times \Pi \to [0,\infty)$. Define $\mathcal{G}(p,q,r) = \frac{d(p,q)+d(q,r)+d(r,p)}{3}$, then $(\Pi,\mathcal{G})$ is a $\mathcal{G}$-metric space.

Definition 2.3[3] A $\mathcal{G}$-metric space is symmetric if $\mathcal{G}(p,q,q) = \mathcal{G}(q,p,p)$, for every $p,q \in \Pi$.

Definition 2.4[3] A sequence $\{p_n\}$ in Π is known to be $\mathcal{G}$-convergent to $u \in \Pi$ if for every $\epsilon > 0$, there always occur a positive integer n_1 so that $\mathcal{G}(p_n,p_m,u) < \epsilon$, for every $m,n > n_1$.

Proposition 2.5[3] In a $\mathcal{G}$-metric space following outcomes are identical:

1. $\{p_n\}$ is convergent,
2. $\mathcal{G}(p_n,p_n,u) \to 0$ as $n \to \infty$,
3. $\mathcal{G}(p_n,u,u) \to 0$ as $n \to \infty$.

Definition 2.6[3] A sequence $\{p_n\}$ in Π is known to be $\mathcal{G}$-Cauchy if for every $\epsilon > 0$, there always occur a positive integer n_0 so that $\mathcal{G}(p_n,p_m,p_m) < \epsilon$, for every $m,n > n_0$.

Proposition 2.7[3] In a $\mathcal{G}$-metric space, following outcomes are same:

1. sequence $\{p_n\}$ is Cauchy,
2. for any $\epsilon > 0$, there always occur a positive integer n so that $\mathcal{G}(p_n,p_m,p_l) < \epsilon$, for all $n,m \in \mathcal{N}$.

Definition 2.8[3] A $\mathcal{G}$-metric space is called complete if every $\mathcal{G}$-Cauchy sequence is convergent in Π.

Proposition 2.9[3] In a $\mathcal{G}$-metric space, following results hold for every $p,q,r,\alpha \in \Pi$:

1. if $\mathcal{G}(p,q,r) = 0$, then $p = q = r$,
2. $\mathcal{G}(p,q,r) \leq \mathcal{G}(p,p,q) + \mathcal{G}(p,p,r)$,
3. $\mathcal{G}(p,q,q) \leq 2\mathcal{G}(q,p,p)$,
4. $\mathcal{G}(p,q,r) \leq \mathcal{G}(p,\alpha,r) + \mathcal{G}(\alpha,q,r)$,

5. $\mathcal{G}(p, q, r) \leq \frac{2}{3}\{\mathcal{G}(p, \alpha, \alpha) + \mathcal{G}(q, \alpha, \alpha) + \mathcal{G}(r, \alpha, \alpha)\}$.

Proposition 2.10[3] In a $\mathcal{G}$-metric space, following results are identical:

1. $(\Pi, \mathcal{G})$ is symmetric,
2. $\mathcal{G}(p, q, r) \leq \mathcal{G}(p, q, w) + \mathcal{G}(r, q, u)$,
3. $\mathcal{G}(p, q, q) \leq \mathcal{G}(p, q, w)$.

3. MAIN RESULTS

Theorem 3.1: Consider $(\Pi, \mathcal{G})$ be a complete $\mathcal{G}$-metric space and let $\mathfrak{I}$ be self-mapping on Π satisfying the following conditions

$$\mathcal{G}(\mathfrak{I}p, \mathfrak{I}q, \mathfrak{I}r) \leq \Phi\mathcal{G}(p.q, r) + \Psi\frac{\mathcal{G}(p,\mathfrak{I}p,\mathfrak{I}p)\mathcal{G}_c(p,\mathfrak{I}q,\mathfrak{I}r)+\mathcal{G}(q,\mathfrak{I}q,\mathfrak{I}r)\mathcal{G}(q,\mathfrak{I}p,\mathfrak{I}p)}{\mathcal{G}(p,\mathfrak{I}q,\mathfrak{I}r)+\mathcal{G}(q,\mathfrak{I}p,\mathfrak{I}p)} \quad (1)$$

for every $p.q, r \in \Pi$ with $\mathcal{G}(p, \mathfrak{I}q, \mathfrak{I}r) + \mathcal{G}(q, \mathfrak{I}p, \mathfrak{I}p) \neq 0$ and $\Phi, \Psi \in \mathcal{R}^+$ with

$\Phi + \Psi < 1$. Then $\mathfrak{I}$ has a unique fixed point.

Proof: Let $p_0 \in \Pi$ be an initial point and $\{p_n\}$ be a sequence in $\mathcal{S}$ and we can choose p_1 in Π such that

$p_1 = \mathfrak{I}p_0, p_2 = \mathfrak{I}p_1 \dots p_{n+1} = \mathfrak{I}p_n$, for every $n \in \mathcal{N}$.

Now

$$\begin{aligned}\mathcal{G}(p_n, p_{n+1}, p_{n+1}) &= \mathcal{G}(\mathfrak{I}p_{n-1}, \mathfrak{I}p_n, \mathfrak{I}p_n)\\ &\leq \Phi\mathcal{G}(p_{n-1}, p_n, p_n)\\ &+ \Psi\frac{\mathcal{G}(p_{n-1}, \mathfrak{I}p_{n-1}, \mathfrak{I}p_{n-1})\mathcal{G}(p_{n-1}, \mathfrak{I}p_n, \mathfrak{I}p_n) + \mathcal{G}(p_n, \mathfrak{I}p_n, \mathfrak{I}p_n)(p_n, \mathfrak{I}p_{n-1}, \mathfrak{I}p_{n-1})}{\mathcal{G}(p_{n-1}, \mathfrak{I}p_n, \mathfrak{I}p_n) + \mathcal{G}(p_n, \mathfrak{I}p_{n-1}, \mathfrak{I}p_{n-1})}\\ &\leq \oplus\ \mathcal{G}(p_{n-1}, p_n, p_n) \text{ where } \oplus = \Phi + \Psi < 1.\end{aligned}$$

$$\mathcal{G}(p_n, p_{n+1}, p_{n+1}) \leq \oplus\ \mathcal{G}(p_{n-1}, p_n, p_n) \leq \oplus^2 \mathcal{G}(p_{n-2}, p_{n-1}, p_{n-1}) \leq \cdots \dots \dots \mathcal{G}(p_0, p_1, p_1) \quad (2)$$

Thus for every $m, n \in \mathcal{N}, m < n$, we have

$$\begin{aligned}\mathcal{G}(p_n, p_m, p_m) &\leq \mathcal{G}(p_n, p_{n+1}, p_{n+1}) + \mathcal{G}(p_{n+1}, p_{n+2}, p_{n+2}) + \cdots \dots \dots + \mathcal{G}(p_{m-1}, p_m, p_m)\\ &\leq \oplus^n \mathcal{G}(p_0, p_1, p_1) + \oplus^{n+1} \mathcal{G}(p_0, p_1, p_1) + \cdots \oplus^{m-1} \mathcal{G}(p_0, p_1, p_1)\end{aligned}$$

$$= \frac{\oplus^{n}}{1-\oplus}\mathcal{G}(\wp_0, \wp_1, \wp_1) \to 0 \text{ as } m, n \to \infty. \tag{3}$$

Thus $\{\wp_n\}$is a Cauchy sequence. Completeness property of $(\Pi, \mathcal{G})$ enables us to get a point $\tau \in \Pi$
so that $\lim_{n\to\infty} \wp_n = \tau$.
We want to show that point $\Im\tau = \tau$. Assume this is not possible and there is $\rho \in \Pi$ such that $\mathcal{G}(\tau, \Im\tau, \Im\tau) = \rho$.
$\rho = \mathcal{G}(\tau, \Im\tau, \Im\tau) \le \mathcal{G}(\tau, \wp_{n+1}, \wp_{n+1}) + \mathcal{G}(\wp_{n+1}, \Im\tau, \Im\tau) = \mathcal{G}(\tau, \wp_{n+1}, \wp_{n+1}) + \mathcal{G}(\Im\wp_n, \Im\tau, \Im\tau)$

$$\le \mathcal{G}(\tau, \wp_{n+1}, \wp_{n+1}) + \Phi\mathcal{G}(\wp_n, \tau, \tau) + \Psi\frac{\mathcal{G}(\wp_n,\Im\wp_n,\Im\wp_n)\mathcal{G}(\wp_n,\Im\tau,\Im\tau)+\mathcal{G}(\tau,\Im\tau,\Im\tau)\mathcal{G}(\tau,\Im\wp_n,\Im\wp_n)}{\mathcal{G}(\wp_n,\Im\tau,\Im\tau)+\mathcal{G}(\tau,\Im\wp_n,\Im\wp_n)} \tag{4}$$

Proceeding limit as $n \to \infty$ we have$\mathcal{G}(\tau, \Im\tau, \Im\tau) \le 0$, which is not possible. $\rho = 0$.
Hence, we have $\Im\tau = \tau$.

Uniqueness: Let τ^* be a one more fixed point of $\Im$ and such that $\tau^* \neq \tau$. Then

$$\mathcal{G}(\tau, \tau^*, \tau^*) = \mathcal{G}(\Im\tau, \Im\tau^*, \Im\tau^*) \le \Phi\mathcal{G}(\tau, \tau^*, \tau^*) + \Psi\frac{\mathcal{G}(\tau.,\Im\tau,\Im\tau)\mathcal{G}(\tau.,\Im\tau^*,\Im\tau^*)+\mathcal{G}(\tau^*,\Im\tau^*,\Im\tau^*)\mathcal{G}(\tau^*,\Im\tau,\Im\tau)}{\mathcal{G}(\tau,\Im\tau^*,\Im\tau^*)+\mathcal{G}(\tau^*,\Im\tau,\Im\tau)} \tag{5}$$

And

$$\mathcal{G}(\tau, \tau^*, \tau^*) \le \Phi\,\mathcal{G}(\tau, \tau^*, \tau^*) + \Psi\frac{\mathcal{G}(\tau, \Im\tau, \Im\tau)\mathcal{G}(\tau, \Im\tau^*, \Im\tau^*) + \mathcal{G}(\tau^*, \Im\tau^*, \Im\tau^*)\mathcal{G}(\tau^*, \Im\tau, \Im\tau)}{\mathcal{G}(\tau, \Im\tau^*, \Im\tau^*) + \mathcal{G}(\tau^*, \Im\tau, \Im\tau)}$$
$$< \Phi\mathcal{G}(\tau, \tau^*, \tau^*).$$

A contradiction, therefore $\tau^* = \tau$. Hence uniqueness follows.
Corollary 3.2 Consider a function $\Im: \Pi \to \Pi$ in a complete $\mathcal{G}$-metric Space $(\Pi, \mathcal{G})$ satisfying the following conditions for any $n \in \mathcal{N}$:

$$\mathcal{G}(\Im^n\wp, \Im^n q, \Im^n r) \le \Phi\mathcal{G}(\wp. q, r) + \Psi\frac{\mathcal{G}(\wp, \Im^n\wp, \Im^n\wp)\mathcal{G}(\wp, \Im^n q, \Im^n r) + \mathcal{G}(q, \Im^n q, \Im^n r)\mathcal{G}(q, \Im^n\wp, \Im^n\wp)}{\mathcal{G}(\wp, \Im^n q, \Im^n r) + \mathcal{G}(q, \Im^n\wp, \Im^n\wp)} \tag{6}$$

for every $\wp. q, r \in \Pi$ with, $\mathcal{G}(\wp, \Im^n q, \Im^n r) + \mathcal{G}(q, \Im^n\wp, \Im^n\wp) \neq 0$ and $\Phi, \Psi \in \mathcal{R}^+$ with

$\Phi + \Psi < 1$. Then $\Im$ has a unique fixed point.

Proof: By last result $\tau \in \Pi$ so that $\Im^n \tau = \tau$.

Now $\mathcal{G}(\tau, \Im\tau, \Im\tau) = \mathcal{G}(\Im^n \tau, \Im\Im^n \tau, \Im\Im^n \tau) = \mathcal{G}(\Im^n \tau, \Im^n \Im\tau, \Im^n \Im\tau)$

$$\leq \Phi \mathcal{G}(\tau. \Im\tau, \Im\tau) + \Psi \frac{\mathcal{G}(\tau, \Im^n \tau, \Im^n \tau)\mathcal{G}(\tau, \Im^n \Im\tau, \Im^n \Im\tau) + \mathcal{G}(\Im\tau, \Im^n \Im\tau, \Im^n \Im\tau)\mathcal{G}(\Im\tau, \Im^n \tau, \Im^n \tau)}{\mathcal{G}(\tau, \Im^n \Im\tau, \Im^n \Im\tau) + \mathcal{G}(\Im\tau, \Im^n \tau, \Im^n \tau)}$$

$\leq \Phi \mathcal{G}(\tau. \Im\tau, \Im\tau)$., a contradiction.

Hence $\Im\tau = \tau$ and mapping $\Im$ $h$as a unique fixed point.

Theorem 3.3: Consider a function $\Im: \Pi \rightarrow \Pi$in a complete $\mathcal{G}$-metric space $(\Pi, \mathcal{G})$ satisfying the following condition

$$\mathcal{G}(\Im p, \Im q, \Im r) \leq \Phi \mathcal{G}(p, q, r) + \Psi \frac{\mathcal{G}(q, \Im q, \Im r)[1 + \mathcal{G}(p, \Im p, \Im p)]}{1 + \mathcal{G}(p, q, r)} + \Gamma \frac{\mathcal{G}(q, \Im q, \Im r) + \mathcal{G}(r, \Im p, \Im p)}{1 + \mathcal{G}(q, \Im q, \Im r)\mathcal{G}(r, \Im p, \Im p)} \quad (7)$$

for every $p. q, r \in \Pi$ and $\Phi, \Psi, \Gamma \in \mathcal{R}^+$ with $\Phi + \Psi + \Gamma < 1$. Then $\Im$ has a unique fixed point.

Proof: Let $p_0 \in \Pi$ be an initial point and $\{p_n\}$ be a sequence in Π and we can choose p_1 in Π such that

$p_1 = \Im p_0, p_2 = \Im p_1 \ldots p_{n+1} = \Im p_n$, for every $n \in \mathcal{N}$.

Now

$$\mathcal{G}(p_n, p_{n+1}, p_{n+1}) = \mathcal{G}(\Im p_{n-1}, \Im p_n, \Im p_n)$$

$$\leq \Phi \mathcal{G}(p_{n-1}, p_n, p_n) + \Psi \frac{\mathcal{G}(p_n, \Im p_n, \Im p_n)[1 + \mathcal{G}(p_{n-1}, \Im p_{n-1}, \Im p_{n-1})]}{1 + \mathcal{G}(p_{n-1}, p_n, p_n)}$$

$$+ \Gamma \frac{\mathcal{G}(p_n, \Im p_n, \Im p_n) + \mathcal{G}(p_n, \Im p_{n-1}, \Im p_{n-1})}{1 + \mathcal{G}(p_n, \Im p_n, \Im p_n)\mathcal{G}(p_n, \Im p_{n-1}, \Im p_{n-1})}$$

$$\mathcal{G}(p_n, p_{n+1}, p_{n+1}) \leq \Phi \mathcal{G}(p_{n-1}, p_n, p_n) + \Psi \mathcal{G}(p_n, p_{n+1}, p_{n+1}) + \Gamma \mathcal{G}(p_n, p_{n+1}, p_{n+1})$$

$$\mathcal{G}(p_n, p_{n+1}, p_{n+1}) \leq \oplus\ \mathcal{G}(p_{n-1}, p_n, p_n) \text{ where } \oplus = \frac{\Phi}{1 - \Psi - \Gamma} < 1.$$

By applying the same argument as discussed in (2) and (3) of the result 3.1, we can show that $\{p_n\}$ is a Cauchy sequence. Completeness property of $(\Pi, \mathcal{G})$ enables us to get a point $\tau \in \Pi$ so that $\lim_{n \to \infty} p_n = \tau$.

We want to show that point $\Im\tau = \tau$. Assume this is not possible and there is $\rho \in \Pi$ such that

$$\mathcal{G}(\tau, \Im\tau, \Im\tau) = \rho > 0.$$

$$\rho = \mathcal{G}(\tau, \Im\tau, \Im\tau)\mathcal{G}(\tau, p_{n+1}, p_{n+1}) + \mathcal{G}(p_{n+1}, \Im\tau, \Im\tau) = \mathcal{G}(\tau, p_{n+1}, p_{n+1}) + \mathcal{G}(\Im p_n, \Im\tau, \Im\tau)$$

$$\leq \mathcal{G}(\tau, \wp_{n+1}, \wp_{n+1}) + \Phi\mathcal{G}(\wp_n\tau, \tau) + \Psi\frac{\mathcal{G}(\tau,\Im\tau,\Im\tau)[1+\mathcal{G}(\wp_n,\Im\wp_n,\Im\wp_n)]}{1+\mathcal{G}(\wp_n,\tau,\tau)} + \Gamma\frac{\mathcal{G}(\tau,\Im\tau,\Im\tau)+\mathcal{G}(\tau,\Im\wp_n,\Im\wp_n)}{1+\mathcal{G}(\tau,\Im\tau,\Im\tau)\mathcal{G}(\tau,\Im\wp_n,\Im\wp_n)} \quad (8)$$

Proceeding limit as $n \to \infty$ we have $\rho = \mathcal{G}(\tau, \Im\tau, \Im\tau) \leq 0$, which is not possible. So $\rho = 0$. Hence, we have $\Im\tau = \tau$.

Uniqueness: Let τ^* be a one more fixed point of $\Im$ and such that $\tau^* \neq \tau$. $\mathcal{G}(\tau, \tau^*, \tau^*) =$
$\mathcal{G}(\Im\tau, \Im\tau^*, \Im\tau^*) \leq \Phi\mathcal{G}(\tau, \tau^*, \tau^*) + \Psi\frac{(\tau^*,\Im\tau^*,\Im\tau^*)[1+\mathcal{G}(\tau,\Im\tau,\Im\tau)]}{1+\mathcal{G}(\tau,\tau^*,\tau^*)} + \Gamma\frac{\mathcal{G}(\tau^*,\Im\tau^*,\Im\tau^*)+\mathcal{G}(\tau^*,\Im\tau,\Im\tau)}{1+\mathcal{G}(\tau^*,\Im\tau^*,\Im\tau^*)\mathcal{G}(\tau^*,\Im\tau,\Im\tau)}$ (9)
applying preposition 2.9, we have
$\mathcal{G}_c(\tau, \tau^*, \tau^*) \leq \Phi\mathcal{G}(\tau, \tau^*, \tau^*) + \Gamma\mathcal{G}(\tau^*, \tau^*, \tau) \leq (\Phi + 2\Gamma)\mathcal{G}(\tau, \tau^*, \tau^*)$

A contradiction, therefore $\tau^* = \tau$. Hence uniqueness follows.

Corollary 3.4: Consider a function $\Im: \Pi \to \Pi$ in a complete $\mathcal{G}$-metric space $(\Pi, \mathcal{G})$ satisfying the following condition for any $n \in \mathcal{N}$:

$$\mathcal{G}(\Im^n\wp, \Im^n q, \Im^n r) \leq \Phi\mathcal{G}(\wp, q, r) + \Psi\frac{\mathcal{G}(q,\Im^n q,\Im^n r)[1+\mathcal{G}(\wp,\Im^n\wp,\Im^n\wp)]}{1+\mathcal{G}(\wp,q,r)} + \Gamma\frac{\mathcal{G}(q,\Im^n q,\Im^n r)+\mathcal{G}(r,\Im^n\wp,\Im^n\wp)}{1+\mathcal{G}(q,\Im^n q,\Im^n r)\mathcal{G}(r,\Im^n\wp,\Im^n\wp)}$$

for every $\wp. q, r \in \Pi$ and $\Phi, \Psi, \Gamma \in \mathcal{R}^+$ with $\Phi + \Psi + \Gamma < 1$. Then $\Im$ has a unique fixed point.

Corollary3.5: Consider a function $\Im: \Pi \to \Pi$ in a complete $\mathcal{G}$-metric space $(\Pi, \mathcal{G})$ satisfying the following condition:

$$\mathcal{G}(\Im\wp, \Im q, \Im r) \leq \Phi\mathcal{G}(\wp, q, r) + \Psi\frac{\mathcal{G}(q, \Im q, \Im r)[1 + \mathcal{G}(\wp, \Im\wp, \Im\wp)]}{1 + \mathcal{G}(\wp, q, r)}$$

for every $\wp. q, r \in \Pi$ and $\Phi, \Psi \in \mathcal{R}^+$ with $\Phi + \Psi < 1$. Then $\Im$ has a unique fixed point.

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